Compactlike breathers: Bridging the continuous with the anticontinuous limit

M. Eleftheriou, B. Dey,* and G. P. Tsironis
Department of Physics, University of Crete and Foundation for Research and Technology–Hellas, P. O. Box 2208, 71003 Heraklion, Crete, Greece
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We consider discrete nonlinear lattices characterized by on-site nonlinear potentials and nonlinear dispersive interactions that, in the continuous limit, support exact compacton solutions. We show that the compact support feature of the solutions in the continuous limit persists all the way to the anticontinuous limit. While in the large coupling regime the compact discrete breather solution retains the essential simple cosinelike compacton shape, in the close vicinity of the anticontinuous limit it acquires a spatial shape characterized by a fast stretched exponential decay, preserving thus its essentially compact nature. The discrete compact breathers in the anticontinuous limit are generated through a numerically exact procedure and are shown to be generally stable.

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Intrinsic localized modes or discrete breathers are localized modes in translationally invariant lattices of nonlinear oscillators that are induced as a result of the coexistence of nonlinearity with lattice discreteness [1–6]. Intense work during the last ten years has addressed and resolved in many cases issues regarding their rigorous existence, numerical construction, stability, dynamics, thermodynamics, quantum aspects, and very recently also experimental manifestation in specific materials [7]. One aspect that discrete breathers seem to share in most cases studied so far with linear localized modes appearing in disordered systems is the typical spatial exponential profile giving rise to a characteristic nonlinear localization length [2,4,6]. In the present work we show that this need not be the case in general, and in fact discrete breathers with compactlike support can be constructed provided the dispersive interoscillator interaction becomes nonlinear. These compactlike breathers (CB's) share some of the usual intrinsic localized mode properties, and in particular they can be discrete while, under some circumstances, they can also be mobile. Furthermore, even though in appropriate continuous limits the discrete equations of motion become partial differential equations with compacton solutions [8], we show that CB's are not necessarily the discrete versions of the latter.

We consider a set of coupled nonlinear oscillators with a Hamiltonian in one dimension:

$$H=\sum_n \left(\frac{1}{2}\dot{u}_n^2+\frac{k_1}{2}(u_{n+1}-u_n)^2 + \frac{k_2}{\alpha+1}(u_{n+1}-u_n)^{\alpha+1}+V(u_n)\right), \tag{1}$$

where the exponent α can in general be noninteger, the potential $V(u_n)$ is a nonlinear on-site potential, $u_n(t)\equiv u_n$ is the displacement of the nth unit mass oscillator from its equilibrium position at time $t,\dot{u}_n$ is the corresponding velocity, and k_1,k_2 determine the strengths of the linear and nonlinear nearest neighbor couplings, respectively. The equations of motion for the displacement at site n are

$$\ddot{u}_n-k_1[u_{n+1}+u_{n-1}-2u_n]-k_2[(u_{n+1}-u_n)^\alpha - (u_n-u_{n-1})^\alpha]+V'(u_n)=0. \tag{2}$$

For the analysis that follows we will use primarily three different potentials, the (soft) double well potential given by $V(u_n)=\frac{1}{4}(1-u_n^2)^2$, the "hard" ϕ^4 potential $V(u_n)=\frac{1}{2}u_n^2(1+\frac{1}{2}u_n^2)$, and also the soft Morse potential $V(u_n)=\frac{1}{2}(1-\exp[-u_n])^2$. Furthermore, we will not study the general case of arbitrary exponent α, but deal, as in Refs. [9–11] with a more restricted case, focusing hereafter on $\alpha=3$, which coincides with one of the celebrated Fermi-Pasta-Ulam problem studies.

Let us start our analysis from the vicinity of the anticontinuous limit [2]. In order to construct a nonlinear localized mode of frequency ω_b on the discrete lattice we use the standard procedure starting from a trivial breather of the same frequency at the anticontinuous limit ($k_1=k_2=0$) and analytically continuing the latter to finite couplings. Furthermore, the linear stability of the mode at each coupling value can be obtained through the eigenvalue analysis of the Floquet matrix of the tangent map associated with the map generated by the solution of Eqs. (2) when the latter are evaluated at times that are multiples of the breather period [4]. The resulting spatial breather distributions in a semilogarithmic plot for the three potentials considered are depicted in Fig. 1, where, in each subplot, there is a comparison between conventional exponentially decaying discrete breathers in a lattice with purely linear dispersive coupling term ($k_1\neq 0$ and $k_2=0$) and those with nonlinear coupling, i.e., for $k_1=0$ and $k_2\neq 0$. We note that in all three potential cases the presence of a purely linear dispersive coupling leads to a clear spatial exponential decay, while nonlinear dispersive coupling results in extremely fast spatial decay to zero. This feature of fast decay to zero within a few sites from the central breather site is a feature stemming from the nonlinear dispersive term

*Permanent address: Department of Physics, University of Pune, Pune 411007, India.

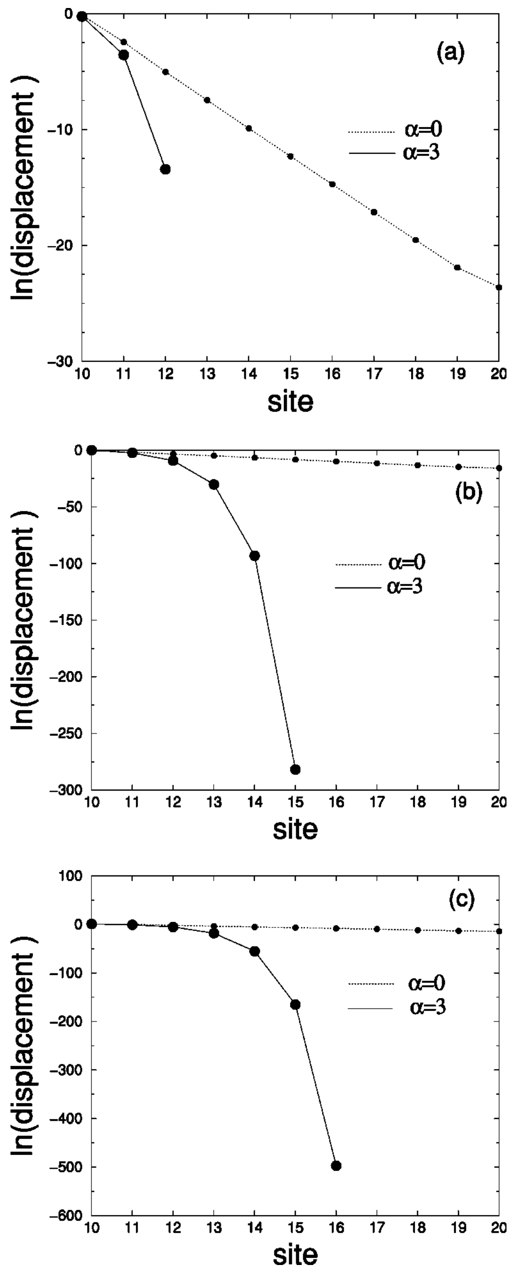

FIG. 1. Spatial compact breather configuration in a semilogarithmic plot. The horizontal axis is the site number, assuming that the central breather site is at $n=10$, while the vertical axis is the logarithm of the oscillation amplitude $\ln(u_n)$. In (a) for the double well potential breather with $\omega_b=1.045$, solid line $k_1=0,k_2=0.1$, dashed line $k_1=0.1,k_2=0$. In (b) hard ϕ^4 potential breather with $\omega_b=1.318$, solid line $k_1=0,k_2=0.1$ (with fitted exponential stretching exponent $s=4.93$), and dashed line $k_1=0.1,k_2=0$. In (c) Morse potential breather with $\omega_b=0.834$, solid line $k_1=0,k_2=0.1(s=5.94)$ and dashed line $k_1=0.1,k_2=0$. For all three nonlinear dispersive cases we show amplitude points that are strictly nonzero even though they are very small.

and is found generically with different nonlinear on-site potentials and coupling constants.

Let us define heuristically a cutoff, than which smaller values can be essentially considered as zero. Let this cutoff number $u_{cut}\approx 10^{-50}$ in dimensionless units. We note that using this cutoff the extent of the breather for nonlinear dispersive coupling is hardly larger than a few sites whereas the

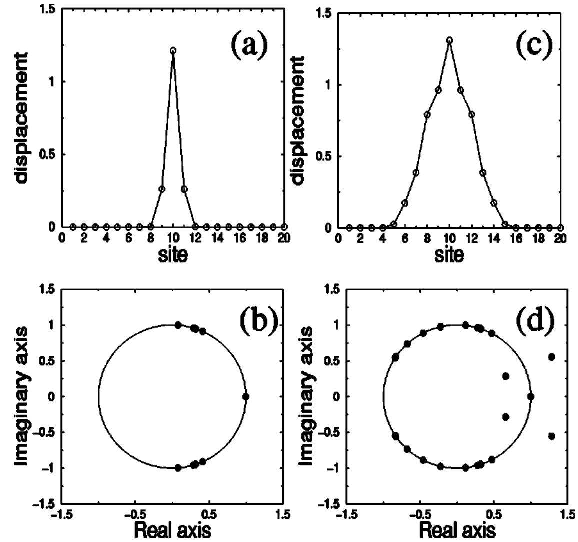

FIG. 2. Initial compact breather shape and stability ($\omega_b=0.834$). In (a) we show the spatial distribution of the compact breather of the Morse potential for $k_1=0,k_2=0.1$, and in (b) the eigenvalues of the Floquet matrix of the tangent map to the CB generation map. They fall on the unit circle and thus the CB solution is linearly stable. In (c) we show the spatial distribution of a linearly unstable compact breather for the same on-site potential for $k_1=0,k_2=4.5$. In (d) two pairs of Floquet eigenvalues of the breather in (c) lie outside the unit circle.

corresponding ones with linear dispersive coupling are much wider. Furthermore, while the exponentially decaying discrete breathers have a long tail, compact breathers are characterized by a dramatic decay and essential absence of any tail. In this sense the breathers generated close to the anticontinuous limit on a lattice with purely nonlinear dispersive interaction can be termed compact. It is important to stress that the amplitude of discrete CB's decays extremely fast with a stretched exponential law $u_n\approx\exp[-\gamma n^s]$ characterized by an exponent $s>4$ for the cases studied. We found using standard fitting procedures that this decay law fits best not only the cases studied in this work but also the earlier data given in Ref. [4] for the pure Fermi-Pasta-Ulam lattice $(s\approx 4)$. On the other hand, this decay law is markedly different from the superexponential decay proposed using approximate arguments for similar problems [11]. The latter functional form not only does not seem to fit the general shape of the compact breather but also clearly misses the initial slope of the decay.

In addition to the generation of CB's from the anticontinuous limit it is also possible to study their linear stability for different values of the coupling constant k_2 of the nonlinear dispersive interaction. In Fig. 2 we present two cases using the Morse potential, for different values of k_2, as well as their associated stability diagrams portraying the Floquet spectrum of eigenvalues of the CB tangent maps. We note that, while for $k_2=0.1$ the CB is stable, in the corresponding case of much larger coupling $k_2=4.5$ the breather is not linearly stable since two pairs of Floquet eigenvalues are lying outside the unit circle. Once these numerically exact CB's are used as initial conditions for the equations of motion Eq. (2), the resulting dynamics portrays the time evolution of the compact breather. In Figs. 3(a) and 3(b) we show

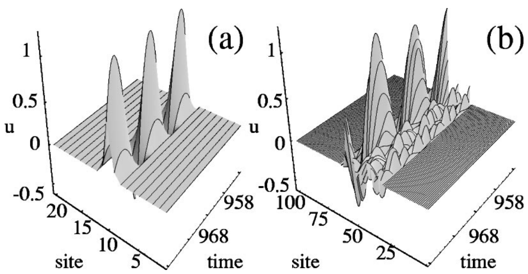

FIG. 3. In (a) we use the compact breather of Fig. 2(a), $k_1 = 0, k_2 = 0.1$, as initial condition for the positions and taking also the initial velocities as zero we find numerically its evolution in time. We note that the breather remains compact at subsequent times ($t = 953 = 126.6T$, where T is the breather period) while no discernible emission of radiation is observed, signifying that the CB is a numerically exact solution of the equations of motion. In (b) we use the compact breather of Fig. 2(c), $k_1 = 0, k_2 = 4.5$, as initial condition for the positions and taking also the initial velocities zero we find numerically its evolution in time. We note the clear presence of radiation that eventually leads to breather decay ($t = 126.6T$).

the time evolution of the two CB's shown in Figs. 2(a) and 2(c) after a reasonably long time from the beginning of the evolution. In the case of Fig. 3(a) we observe that for the linearly stable CB of Fig. 2(a) the compact shape of the solution survives for long times accompanied by a remarkable absence of radiation emission. In the case of the linearly unstable breather of Fig. 2(c), on the other hand, we observe in Fig. 3(b) a slow decay followed by the emission of radiation. We note that similar results have been verified for the cases of the other potentials studied in this work.

Let us now come to the vicinity of the continuous limit. The continuum equation is obtained in the usual way through the approximation of the dispersive terms with a series involving derivatives of the unknown wave amplitude function and truncation of the series in an appropriate order. For the ϕ^4 type of on-site potential, it is possible to derive the exact compacton breather solutions of the continuum equations. To see this, we expand $u_{n\pm1}$ up to fourth order in a Taylor series to get the continuum equation corresponding to Eq. (2) for $\alpha = 3$ and $k_1 = 0$ as

$$\frac{\partial^2 u}{\partial t^2} = 3k_2\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} - u - u^3. \tag{3}$$

Using the ansatz $u(x,t) = G(t)\phi(x)$ for the compact breather solution and substituting it in Eq. (3) above, we get the equations for $\phi(x)$ and $G(t)$, respectively, as

$$3k_2\phi_x^2\phi_{xx} - \phi^3 + C\phi = 0, \tag{4}$$

$$\ddot{G} + G + CG^3 = 0, \tag{5}$$

where C is an arbitrary constant. It can easily be checked that the solutions of Eqs. (4) and (5) give a compacton breather solution for the continuum equation corresponding to Eq. (2) as

$$u(x,t) = \begin{cases} A\cos(Bx)\ \mathrm{cn}\left(\dfrac{(A^2+2)^{1/2}wt}{\sqrt{2}}, k^2\right) \\ \text{for } |Bx| \le \dfrac{\pi}{2} \\ = 0 \ \text{otherwise}, \end{cases} \tag{6}$$

where the inverse width of the compacton $B = (1/3k_2)^{1/4}, k = A/[2(2+A^2)]^{1/2}$, and cn is a Jacobian elliptic function of time. We have also checked that compacton solutions of the same shape as in Eq. (6) (with different A and B) are also allowed when we retain the next higher order dispersion terms in Eq. (3) (sixth order in the Taylor series expansion). Now we consider the case when $k_1 \neq 0$. To get the compacton breather solutions in this case we use the ansatz $u_n(t) = A\phi_n \cos wt$, where w is the frequency of the compacton breather. Expanding $\phi_{n\pm1}$ up to fourth order in the Taylor series and using the rotating wave approximation, we get the continuum equation corresponding to Eq. (2) as

$$k_1\left(\phi_{xx} + \frac{1}{12}\phi_{4x}\right) + \frac{9}{4}k_2A^2\phi_x^2\phi_{xx} - (1+w^2)\phi - \frac{3}{4}A^2\phi^3 = 0. \tag{7}$$

Again, it can easily be checked that the compacton solution to the above equation is given by $\phi(x) = A\cos(Bx)$ for $|Bx| \le \pi/2$ and where $B = (1/6k_2)^{1/4}$ and thus the compacton breather solution in this case is given by $u(x,t) = A\cos(Bx)\cos wt$. We would like to point out that the first derivatives of these solutions are discontinuous at the edge and hence the compacton solutions presented here must be understood in the weak sense. The robustness of these compacton solutions is yet unknown. However, as reported by Rosenau (see Ref. [12] in [8(b)]), extensive numerical studies of the continuum equations indicate that compacton smoothness at the edge is not indicative of stability.

Recent work [12] has shown that a class of exact continuous compacton solutions of a low order continuous approximation of the discrete equations survive in general when substituted in the discrete equations of motion of Eq. (2). What is remarkable is that according to Ref. [12] the continuous cosinelike compacton solutions seem to represent quite well breather solutions in highly discrete regimes far from the continuous limit. One obvious question then relates to the connection between the discrete cosinelike solutions of Ref. [12] and the single breathers obtained here through generation from the anticontinuous limit. While the latter do not have a cosine shape, they do indeed have a bounded support, in the sense described previously. Furthermore, they are in general quite stable for very long times. The cosinelike discrete breathers, on the other hand, while describing the proper bounded solutions in the regime close to the continuous limit, do not seem to have the feature of long-term stability in the opposite limit, even though for relatively short times they too seem to be quite stable. For instance, in the case of a CB for the hard potential $V(u_n) = 4(100\frac{1}{2}u_n^2 + \frac{1}{4}u_n^4)$ having a width of 57 sites [12], peak amplitude 0.1, and period $T \approx 0.3$, we found through simulations that it starts losing its shape after approximately 100 periods of oscillation. While it does not seem to collapse at these times, nevertheless it develops distortions in the tail. In general, however, we also find that the essential effect of the nonlinear dispersive interaction is to introduce a truncation mechanism leading to more compact localized solutions with small or negligible tails. Thus, compact solutions are favored also in the continuous limit and can survive quite well for short times even if they are not the exact discrete solutions.

Another outstanding issue regarding discrete breathers in general and thus also discrete CB's is their mobility. In recent work, it has been shown numerically that it is possible to excite breathers in soft on-site potentials through antisymmetric linear modes and thus render them mobile [5]. While this approach does not provide exact moving breather solutions, if there are any, it nevertheless demonstrates clearly and systematically mobility properties of discrete breathers. In the case of CB's, one expects reduced mobility capabilities since the discontinuity between the excited and nonexcited lattice sites necessitates substantial initial depinning energy that quickly destroys the compact breather. As a result, discrete CB's are much more immobile than the usual exponentially decaying discrete breathers of soft nonlinear potentials. A numerical search for the case of the Morse potential discrete CB's has shown some traces of mobility, which is, however, substantially reduced compared to that of conventional Morse discrete breathers. While discrete CB's can move, their motion has a short duration and also does not seem to be as uniform as that of the usual discrete breathers. More work in this direction is necessary.

Finally, let us discuss the case of the combined presence of both linear and nonlinear dispersive coupling. As has also been noted elsewhere [11,12], linear coupling does indeed overtake nonlinear and thus the resulting breathers are not compact. However, this change from a compact to an exponential decaying breather does not happen automatically as soon as we include linear dispersive coupling. In Fig. 4 we show on a semilogarithmic scale the spatial decay of discrete breathers close to the anticontinuous limit with simultaneous nonlinear and linear coupling. We note that when linear coupling is sufficiently small the discrete breather becomes progressively more extended, although it does not acquire immediately an exponentially decaying shape. For larger k_1 values the linear dispersive term overtakes the nonlinear one and the resulting breather has the standard exponential tail.

In closing, compactlike breathers exist in the whole range from strong to intermediate to the weak coupling regime provided the interaction between the nonlinear oscillators is

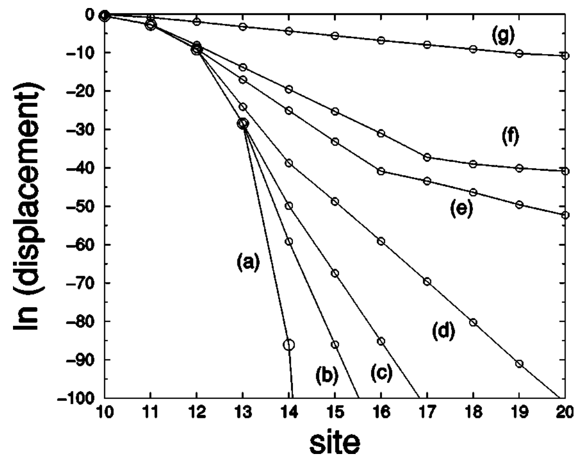

FIG. 4. Combined action of k_1 and k_2 springs. Spatial breather configuration in a semilogarithmic plot. The horizontal axis is the site number, assuming that the central breather site is at $n=10$, while the vertical axis is the logarithm of the oscillation amplitude $\ln(u_n)$. All the cases are for a Morse on-site potential and for a breather with $\omega_b=0.919$. $k_2=0.1$ while for k_1 we have (a) $k_1=0$, (b) $k_1=5\times10^{-15}$, (c) $k_1=5\times10^{-11}$, (d) $k_1=5\times10^{-7}$, (e) $k_1=5\times10^{-5}$, (f) $k_1=5\times10^{-4}$. In (g) $k_1=0.1, k_2=0$.

nonlinear and dispersive. While in the continuous limit the compact breathers are exact compacton solutions with strict bounded support, in the other extreme of the anticontinuous limit the compactlike breathers are characterized by a very fast stretched exponential decay giving rise to an essentially bounded support. These discrete compact breathers are generally stable and in some cases also show mobility properties. In the intermediate nonlinear coupling regime, breathers with bounded support also exist but their stability and shape depend strongly on the parameter regime and breather frequency. The inclusion of linear dispersive coupling additionally to the nonlinear coupling progressively turns compact breathers into conventional exponentially decaying ones. We note that compact breathers would be ideal for energy storage since due to the lack of an exponential tail they interact extremely weakly with each other, thus substantially increasing their lifetime.

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